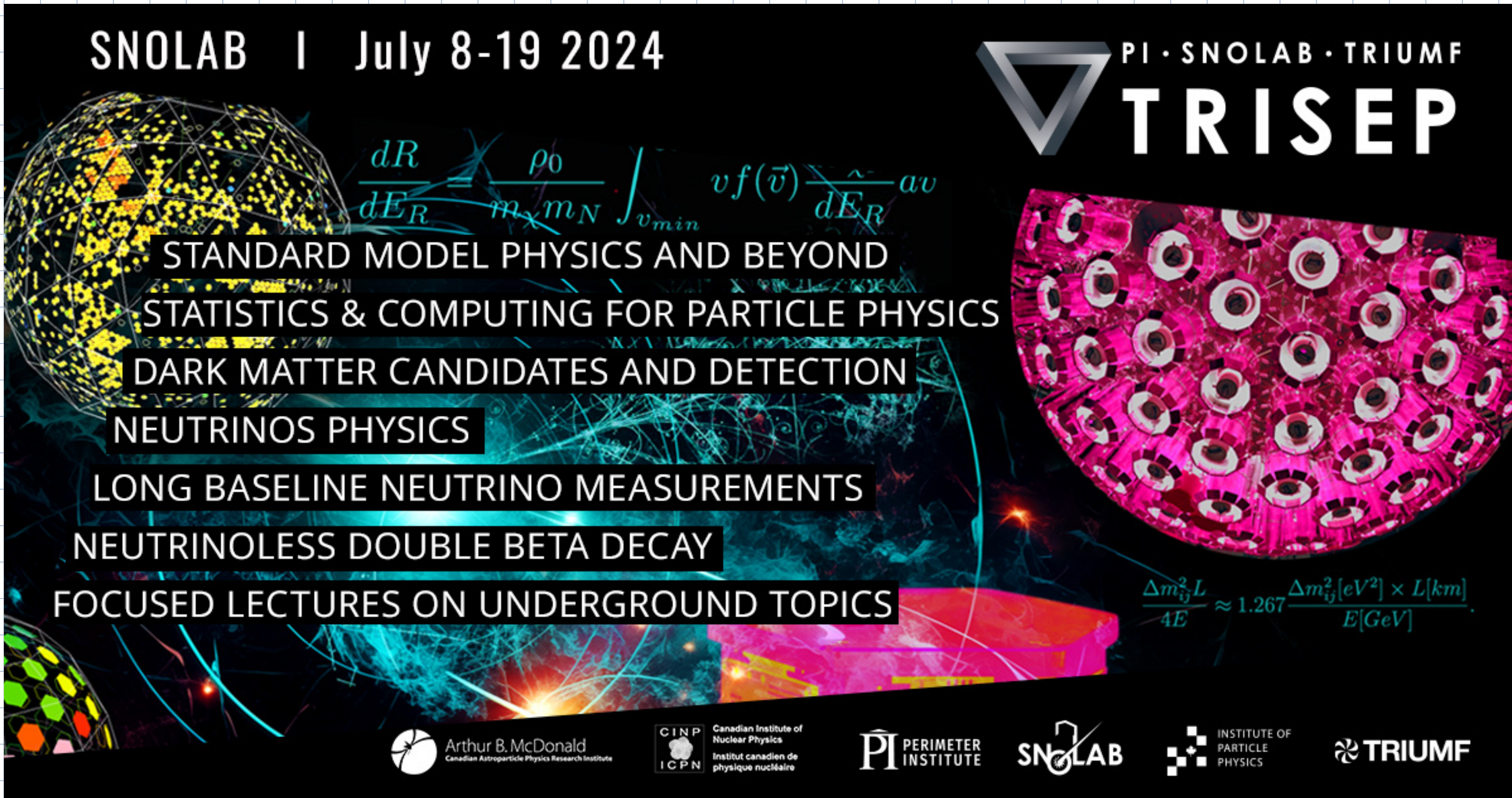

The Standard Model

Problem Solutions

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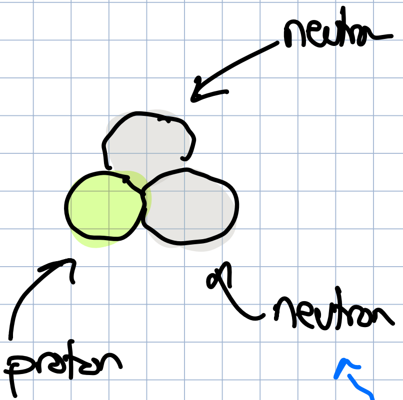
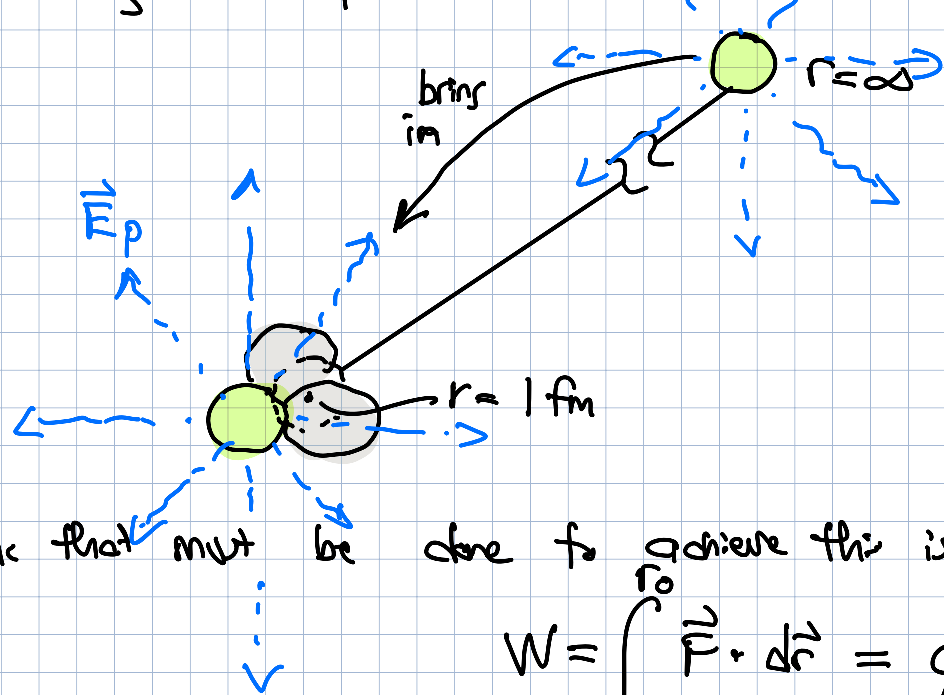
Problem 1.1

I have a tritium nucleus:



I want to bring in a proton from $r = \infty$:



The work that must be done to achieve this is:

$$W = \int_{\infty}^{r_0} \vec{F} \cdot d\vec{r} = q\,\Delta V$$

$$= q\left(V_{r_0} - V_{\infty}\right)$$

$r_0 = 1$ fm from the centre of the original proton

$V_{\infty} = \emptyset$, and $V_{r_0} = k\dfrac{q}{r_0}$ where $q = e = 1.602\times10^{-19}\,C$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^{9}\ \frac{N m^2}{C^2}$$

$$W = e^2 k / r_0 = (1.602\times10^{-19}\,C)^2\left(8.99\times10^{9}\,\frac{Nm^2}{C^2}\right)/(1\times10^{-15}\,m)$$

$$= 2.307\times10^{-13}\ J$$

$1.602\times10^{-19}\ J = 1\,eV \implies W = 1.440\times10^{6}\ eV \approx \boxed{1\ MeV}$

Comment: this is absurdly consistent with the observed energy scale of nuclear phenomena!

BONUS: the wavelength of a photon emitted from a nuclear process would correspond to this ~ 1 MeV energy scale

$$E = hf = 2\pi\hbar f \qquad c = \lambda f \implies E = 2\pi \frac{\hbar c}{\lambda}$$

$$\hbar \approx 1.504 \times 10^{-34} \text{ J}\cdot\text{s}$$

or (from pdg.lbl.gov, "Physical Constants")

Planck constant	h	$6.626\ 070\ 15 \times 10^{-34}$ J s (or J/Hz) §	exact
Planck constant, reduced	$\hbar \equiv h/2\pi$	$1.054\ 571\ 817\ldots \times 10^{-34}$ J s	exact*
		$= 6.582\ 119\ 569\ldots \times 10^{-22}$ MeV s	exact*

$$6.582 \times 10^{-22} \text{ MeV}\cdot\text{s}$$

$c = 2.998 \times 10^{25}$ fm/s

fm $= 10^{-15}$ m

$$\hbar c \approx 197 \text{ MeV}\cdot\text{fm}$$

$$\hookrightarrow \lambda = \frac{2\pi(197 \text{ MeV}\cdot\text{fm})}{1 \text{ MeV}}$$

$$= 2\pi\ 197 \text{ fm} = 1238 \text{ fm}$$

$$\approx 1 \text{ pm}$$

A picometre wavelength is a gamma ray.

Problem 1.2

Convert the electron rest mass in MKS $\rightarrow$ MeV/c^2

using $E^2 = p^2c^2 + m^2c^4 \xrightarrow[\text{at rest}]{} E = mc^2$

mass

$E = mc^2 = 9.109 \times 10^{-31}\ \text{kg} \times (2.998 \times 10^{8}\ \text{m/s})^2$

Mass-energy

$= 8.187 \times 10^{-14}\ \text{J} \xrightarrow{\text{eV}} 511.1 \times 10^{3}\ \text{eV}$

so $m_e = \dfrac{E}{c^2} = \boxed{0.5111\ \text{MeV}/c^2}$

Problem 1.3

Treat β-decay of free neutron as if the process proceeds by:

$$n^0 \rightarrow p^+ + e^-$$

treat n^0 as being at rest. then:

$$p^+ \longleftarrow n^0 \longrightarrow e^-$$

$\vec{p}_p$ $\quad$ $\vec{p}_{n^0} = \emptyset$ $\quad$ $\vec{p}_e$

E_p $\quad$ $E_{n^0} = m_n c^2$ $\quad$ E_e

Recall that energy + momentum are separately conserved;

$E_n = E_p + E_e \longrightarrow m_n c^2 = E_p + E_e$

$\vec{p}_n = \vec{p}_p + \vec{p}_e \longrightarrow \emptyset = \vec{p}_p + \vec{p}_e \longrightarrow \vec{p}_p = -\vec{p}_e$

$|\vec{p}_p| = |\vec{p}_e|$

And in total:

$$\mathbb{P}_n = \mathbb{P}_p + \mathbb{P}_e \qquad \text{where} \left\{ \begin{array}{l} \mathbb{P} = (E, \vec{p}c) \\ \mathbb{P}^2 = E^2 - \vec{p}\cdot\vec{p}\,c^2 \\ \quad = m^2c^4 \end{array} \right| \text{4-vectors!}$$

$$\mathbb{P}_e = \mathbb{P}_n - \mathbb{P}_p$$

$$\mathbb{P}_e^2 = (\mathbb{P}_n - \mathbb{P}_p)^2$$

$$m_e^2 = \mathbb{P}_n^2 - 2\mathbb{P}_n\mathbb{P}_p + \mathbb{P}_p^2 = m_n^2c^4 - 2\mathbb{P}_n\mathbb{P}_p + m_p^2c^4$$

$$m_e^2 = m_n^2c^4 + m_p^2c^4 - 2\Big(E_nE_p - \underbrace{\vec{p}_n\cdot\vec{p}_p c^2}_{\text{vanishes, } \vec{p}_n = \emptyset}\Big)$$

$$= m_n^2c^4 + m_p^2c^4 - 2m_nc^2E_p$$

so

$$E_p = \frac{m_n^2 + m_p^2 - m_e^2}{2m_nc^2}c^4$$

Then:

$$E_e = E_n - E_p = m_nc^2 - \frac{m_n^2 + m_p^2 - m_e^2}{2m_nc^2}c^4$$

$$= \frac{2m_n^2c^4}{2m_nc^2} - \frac{m_n^2 + m_p^2 - m_e^2}{2m_nc^2}c^4$$

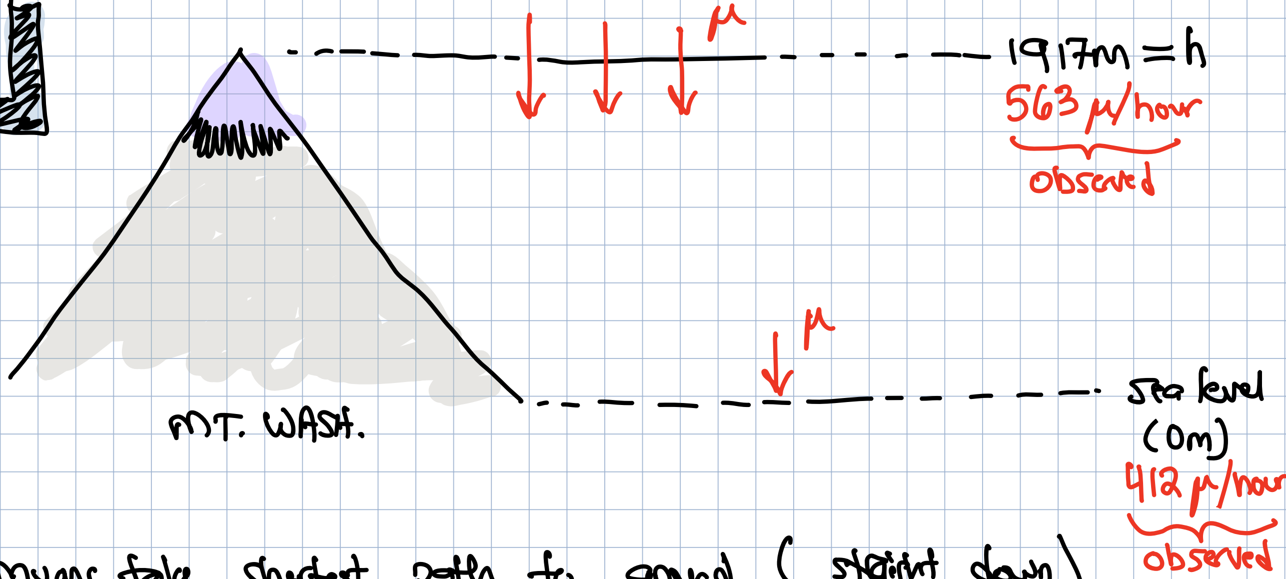
$$\boxed{E_e = \frac{m_n^2 - m_p^2 + m_e^2}{2m_nc^2}c^4}$$

This yields a <u>single</u> value for E_e determined from;

$$\left.\begin{array}{l} m_n = 939.56542052 \text{ MeV}/c^2 \\ m_p = 938.27208943 \text{ MeV}/c^2 \\ m_e = 0.51099895069 \text{ MeV}/c^2 \end{array}\right\} \boxed{E_e = 1.2926\ldots \text{ MeV}}$$

Which looks nothing like a real β-decay spectrum.

Problem 1.4



assume muons take shortest path to ground (straight down).
at h, $N_\mu = 563$ muons (in 1 hour).

Classical calculation

For $\tau = 2.1969\ldots\ \mu s$ lifetime, each $\Delta t = \tau$ results in a reduction of the survival probability by $1/e = 0.367\ldots$

What is the probability of survival for muons if we ignore relativity?

↳ how long does it take muons to cross the 1917m gap?

$$\Delta y = h = v_\mu \Delta t \rightarrow \Delta t = \frac{h}{v_\mu}$$

$$\Delta t_{min} \text{ when } v_\mu = 0.9954c \Rightarrow \frac{h}{v_\mu^{max}} = 6.4238 \times 10^{-6}\ s$$

$$\Delta t_{max} \text{ when } v_\mu = 0.9950c \Rightarrow \frac{h}{v_\mu^{min}} = 6.4264 \times 10^{-6}\ s$$

$$P^{min}_{survival} = e^{-\Delta t_{max}/\tau} = 0.05366$$

$$P^{max}_{survival} = e^{-\Delta t_{min}/\tau} = 0.05372$$

so classically, $N_{sealevel} = N_h \times \begin{cases} P^{max}_{survival} \\ P^{min}_{survival} \end{cases} = \begin{matrix} 30.246 \\ 30.210 \end{matrix}$

$$\boxed{\langle N_{sealevel} \rangle = 30.228 \pm 0.018} \quad \text{(classical)}$$

This is way below the observation.

Relativistic Calculation

We can use either of two approaches:

1) Muon's perspective: it measures proper length, and observes the height h to be contracted by $h' = h/\gamma$

where $\gamma = (1-\beta^2)^{-1/2}$

2) earth observer perspective: the muon's clock appears to run slowly, so its lifetime is dilated to $\tau' = \gamma\tau$

What is γ?

$$\gamma_{min} = (1-\beta^2_{min})^{-1/2} = 10.013$$

$$\gamma_{max} = (1-\beta^2_{max})^{-1/2} = 10.438$$

I will use the second approach — it's a little faster. Try the first approach to confirm the results are the same.

$$P^{min}_{survival} = e^{-\Delta t_{max}/\gamma_{min}\tau} = 0.7467$$

$$P^{max}_{survival} = e^{-\Delta t_{min}/\gamma_{max}\tau} = 0.7557$$

$$\langle N_{sealevel} \rangle = N_\mu \frac{P_{max} + P_{min}}{2} = \boxed{422.9 \pm 2.5}$$ relativity

The real experiment observed 412 μ/hour. Applying Poisson errors to the count, $412 \pm \sqrt{412} = 412 \pm 20$, we find remarkable agreement:

$$\boxed{\begin{aligned} \langle N_\mu \rangle_{expected} &= 422.9 \pm 2.5 \\ N_\mu^{observed} &= 412 \pm 20 \end{aligned}}$$

wow!

Problem 1.5

See solution in Python in Github. I used zfit to model the unbinned data using

$$P(t) = Ae^{-\lambda t} + B$$

$$\tau = 1/\lambda$$

Problem 2.1

We want a simple method that estimates the mass of a hypothetical omega (Ω) baryon.

We are told:

$$m_\Delta \sim 1232 \text{ MeV}/c^2 \qquad (\Delta \in \Delta^-, \Delta^0, \Delta^+, \Delta^{++})$$

$$m_{\Sigma^*} \sim 1385 \text{ MeV}/c^2 \qquad (\Sigma^* \in \Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+})$$

$$m_{\Xi^*} \sim 1533 \text{ MeV}/c^2 \qquad (\Xi^* \in \Xi^{*-}, \Xi^{*0})$$

are the "mass splittings" between neighboring states in the incomplete decuplet the same, or similar?

$$m_{\Sigma^*} - m_\Delta = 1385 - 1232 = 153 \text{ MeV}/c^2$$

$$m_{\Xi^*} - m_{\Sigma^*} = 1533 - 1385 = 148 \text{ MeV}/c^2$$

huh. remarkably similar!

$$\langle \Delta m \rangle = \frac{153 + 148}{2} = 150.5 \text{ MeV}/c^2$$

so estimate $m_\Omega = m_{\Xi^*} + \langle \Delta m \rangle$

$$= 1533 + 151 = \boxed{1684 \text{ MeV}/c^2}$$

The real observed mass is $m_\Omega^{obs} = 1672 \text{ MeV}/c^2$.
This was a remarkable guess, and definitely hinted at an underlying rule.
The quark picture tells us that, for example, $\Xi^{*0} = \bar{s}\bar{s}\bar{u}$ (or ssu)

and $\Omega^+ = \bar{s}\bar{s}\bar{s}$ $\rightarrow$ we are just adding a strange quark ($m_s \sim 95$ MeV) in place of an up quark ($m_u \sim 2.2$ MeV), increasing the mass with a proportional response. The binding energy is also affected by having $\underbrace{\left(-\tfrac{1}{3}e + -\tfrac{1}{3}e + \tfrac{2}{3}e\right)}_{\text{"dipolar" less potential energy.}}$ with $\underbrace{\left(-\tfrac{1}{3}e + -\tfrac{1}{3}e + -\tfrac{1}{3}e\right)}_{\text{purely repulsive, more potential energy}}$

Problem 2.2

- Estimate the branching fraction of $W^+ \rightarrow$ leptons. The available doublets are:

count states $\downarrow$

$u\bar{d}$	$c\bar{s}$	$e^+\nu_e$	$\mu^+\nu_e$	$\tau^+\nu_\tau$
1 charge state	1 charge state	1 charge state each		
3 colour states	3 colour states	0 colour states each		
3 unique final states	3 unique final states	1 unique final state	1 unique final state	1 unique final state

so all available final states, without violating conservation of energy or using virtual particles, add up to $\underbrace{3+3}_{\text{hadronic}} + \underbrace{1+1+1}_{\text{leptonic}} = 9$ final states.

$$B_{\text{lepton}} = \frac{\text{leptonic states}}{\text{all states}} = \frac{3}{9} = \boxed{30\%.}$$

we then expect $\mathcal{B}_{hadrons} \approx 70\%$. Look at PDG to see:

W^+ **DECAY MODES**

W^- modes are charge conjugates of the modes below.

	Mode		Fraction (Γ_i/Γ)	Scale Factor/ Conf. Level	P(MeV/c)
Γ_1	$\ell^+\nu$	[1]	$(10.86 \pm 0.09)\%$		
Γ_2	$e^+\nu$		$(10.71 \pm 0.16)\%$		40185
Γ_3	$\mu^+\nu$		$(10.63 \pm 0.15)\%$		40185
Γ_4	$\tau^+\nu$		$(11.38 \pm 0.21)\%$		40165
Γ_5	hadrons		$(67.41 \pm 0.27)\%$		

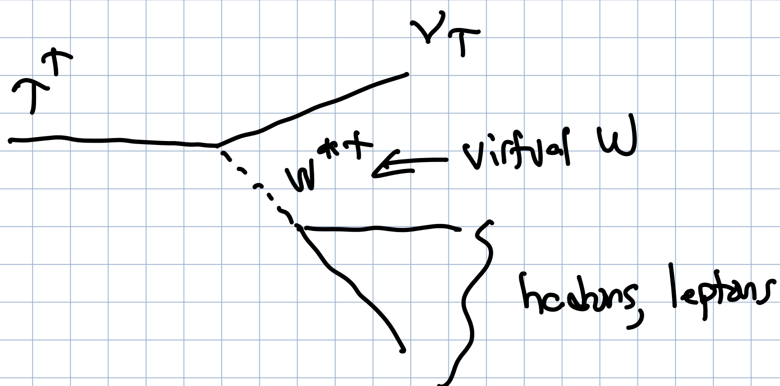
$$\underbrace{\Gamma_2 + \Gamma_3 + \Gamma_4}_{\text{observed}} = (10.71 + 10.63 + 11.38)\%$$

$$= 32.72\% \approx \underline{33\%}$$

$\Rightarrow$ pretty close to expectation from W universality!

- Predict the branching fraction of $\tau^+ \to$ leptons.

$M_\tau = 1.78\ \mathrm{GeV}/c^2 \longrightarrow W^+ \to c\bar{s}$ makes hadrons in final state.



Lightest $c\bar{s}$ meson: D_s^+, $\underbrace{1968\ \mathrm{MeV}/c^2}_{> m_\tau!}$

Lightest charm mesons: $\underbrace{D^\pm, D^0}_{\sim 1900\ \mathrm{MeV}/c^2,\ > m_\tau!}$

$\tau^+ \to c\bar{s}\,\nu_\tau$ forbidden by energy conservation!

we state count again, adjusting for forbidden final states

$\Downarrow$ see next page.

count states

$u\bar{d}$	~~$c\bar{s}$~~	$e^+\nu_e$	$\mu^+\nu_e$	~~$\tau^+\nu_\tau$~~
1 charge state	~~1 charge state~~	1 charge state each		
3 colour states	~~3 colour states~~	0 colour states each		
3 unique final states	~~3 unique final states~~	1 unique final state	1 unique final state	~~1 unique final state~~

real τ cannot decay to itself

$\tau \to$ anything : $3 + 1 + 1 = 5$

$$B_{\tau \to \text{leptons}} = \frac{2}{5} = 0.4 \quad (40\%)$$

$$B_{\tau \to \text{hadrons}} = \frac{3}{5} = 0.6 \quad (60\%)$$

compare to data :

Γ_3	$\mu^-\bar{\nu}_\mu\nu_\tau$	[1]	$(17.39 \pm 0.04)\%$	885	⌄
Γ_4	$\mu^-\bar{\nu}_\mu\nu_\tau\gamma$	[2]	$(3.67 \pm 0.08) \times 10^{-3}$	885	⌄
Γ_5	$e^-\bar{\nu}_e\nu_\tau$	[1]	$(17.82 \pm 0.04)\%$	888	⌄
Γ_6	$e^-\bar{\nu}_e\nu_\tau\gamma$	[2]	$(1.83 \pm 0.05)\%$	888	⌄
Γ_7	$h^- \geq 0K^0_L\ \nu_\tau$		$(12.03 \pm 0.05)\%$	883	⌄
Γ_8	$h^-\nu_\tau$		$(11.51 \pm 0.05)\%$	883	⌄
Γ_9	$\pi^-\nu_\tau$	[1]	$(10.82 \pm 0.05)\%$	883	⌄
Γ_{10}	$K^-\nu_\tau$	[1]	$(6.96 \pm 0.10) \times 10^{-3}$	820	⌄
Γ_{11}	$h^- \geq 1$ neutrals ν_τ		$(37.00 \pm 0.09)\%$		⌄
▾ Modes with three charged particles					
Γ_{62}	$h^-h^-h^+ \geq 0$ neutrals $\geq 0K^0_L\ \nu_\tau$		$(15.20 \pm 0.06)\%$	861	⌄

$\Gamma_3 + \Gamma_5 = 17.39\% + 17.82\% = 35.21\% \approx 35\%$

$\Gamma_8 + \Gamma_{11} + \Gamma_{62} = 11.51\% + 37\% + 15.2\% = 63.71\%$
$\approx 64\%$

$99\% \approx$

So compare 35% (observed) to 40% (estimated) $\longrightarrow$
not bad!

The simple requirements imposed by the SM makes for
powerful constraints.