A deeper look at QTNM

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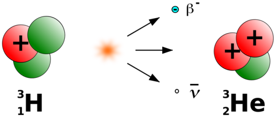
Format of this session

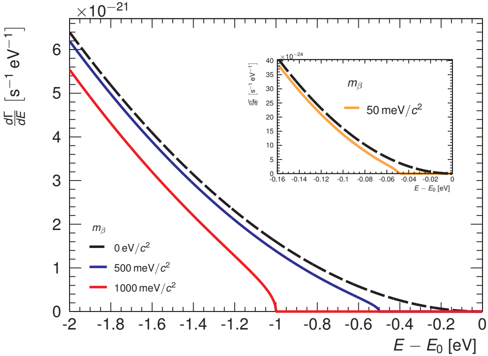
- A closer look at some of the numbers in QTNM
- I'll pose a few questions around some of the key QTNM concepts, you'll give answering them a go and then we'll talk through the solutions
- *Hopefully*, this will give you a bit of an insight into some of the design choices to do with a CRES experiment

Direct measurements of neutrino mass

Measure β-decay electron energy spectrum of tritium



$$m_\beta = \sqrt{\sum_{i=1}^{3} |U_{ei}|^2 m_i^2}$$

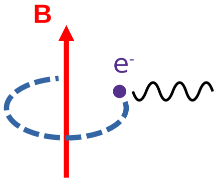


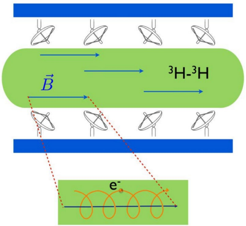
Information about neutrino masses, m_β, encoded close to endpoint of spectrum – $E_0 \approx 18.6\,\text{keV}$

CRES overview

- **C**yclotron **R**adiation **E**mission **S**pectroscopy
- Concept pioneered by Project 8 collaboration[a]
- β-decay electrons immersed in B-field emit EM radiation – frequency depends only on electron energy and B-field strength
- $E_{\text{kin}} \approx Q_\beta \approx 18.6\,\text{keV}$
- Radiation collected with antenna, waveguide or resonant cavity

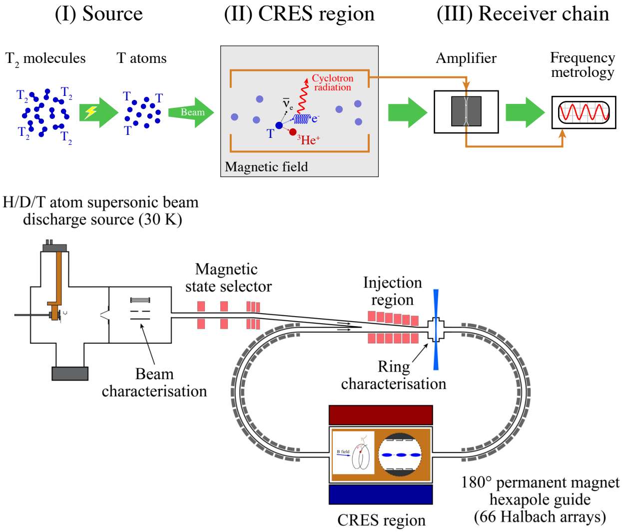
$$f = \frac{1}{2\pi} \frac{eB}{m_e + E_{\text{kin}}/c^2}$$





[a] B. Monreal and J. A. Formaggio, Phys. Rev. D **80** (2009).

Outline of QTNM

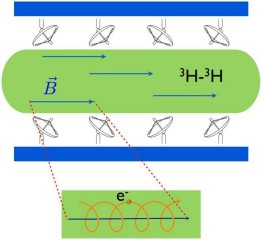


What kind of frequencies do we expect to detect?

- Frequency of cyclotron radiation given by

$$\nu_{\text{cyc}} = \frac{1}{2\pi}\frac{eB}{\gamma m_e} = \frac{1}{2\pi}\frac{eB}{m_e + E_{\text{kin}}/c^2}$$

- A tritium endpoint electron has $E_{\text{kin}} \approx 18.6\,\text{keV}$.
- Choose the magnetic field to be $B = 1\,\text{T}$



Solution

For $E_{\text{kin}} = 18.6\,\text{keV}$ and with $m_e = 511\,\text{keV}/c^2 = 9.11 \times 10^{-31}$ kg we find that $\gamma = 1.0364$.

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From that:

$$\nu_{\text{cyc}} = \frac{1}{2\pi}\frac{1.602 \times 10^{-19}\,\text{C} \cdot 1\,\text{T}}{1.0364 \cdot 9.11 \times 10^{-31}\,\text{kg}}$$

$$\nu_{\text{cyc}} \approx 27\,\text{GHz}$$

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$$\nu_{\mathrm{cyc}} \approx 27\,\mathrm{GHz}$$

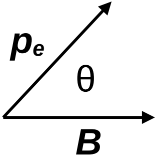
- This is equivalent to a wavelength of $\sim 1\,\mathrm{cm}$
- So we are looking at microwave radiation – detect with antennas, waveguides or resonant cavities

How much power do we detect from each electron?

Total power radiated by a charge in a magnetic field given by

$$P_{\text{Larmor}} = \frac{2\pi e^2 \nu_{\text{cyc}}^2 \beta^2 \sin^2\theta}{3\epsilon_0 c}$$

θ is the 'pitch angle' between the magnetic field direction and the electron's momentum vector.



Solution

For $\gamma = 1.036$, $\beta = 0.26$ and we find that $P_{\text{Larmor}} \sim 1$ fW for $B = 1$ T and $\theta = \frac{\pi}{2}$.

Solution

For $\gamma = 1.036$, $\beta = 0.26$ and we find that $P_{\text{Larmor}} \sim 1\,\text{fW}$ for $B = 1$ T and $\theta = \frac{\pi}{2}$.

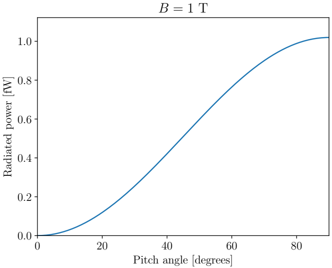
Small radiated power necessitates a strong magnetic field and amplifiers with very low noise

Radiated powers

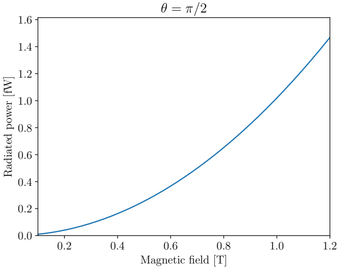
$$P_{\text{Larmor}} = \frac{2\pi e^2 \nu_{\text{cyc}}^2 \beta^2 \sin^2\theta}{3\epsilon_0 c}$$

What does this equation tell us about how to design our experiment for optimum signal collection?

Solution



Those electrons with pitch angles close to 90° are the most detectable



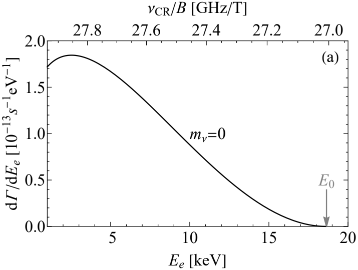
$$P_{\text{Larmor}} \propto B^2$$

High fields advantageous for power detection

Experimental bandwidths

Say we want to measure the last 100 eV of the β-decay spectrum in a 1 T field.

$$\nu_{\text{cyc}} = \frac{1}{2\pi}\frac{eB}{\gamma m_e} = \frac{1}{2\pi}\frac{eB}{m_e + E_{\text{kin}}/c^2}$$

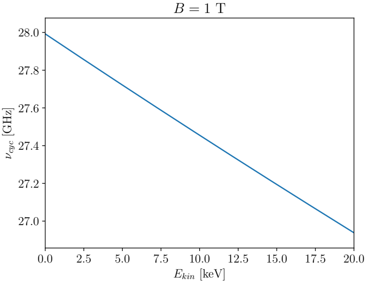


What kind of bandwidth do our experimental components need to have to cover this?

Solution

$$\nu_{\text{cyc}} = \frac{1}{2\pi}\frac{eB}{m_e + E_{\text{kin}}/c^2} = \frac{eBc^2}{2\pi}\left(E_{\text{tot}}\right)^{-1}$$

Entire decay spectrum covers a frequency range of about 1 GHz



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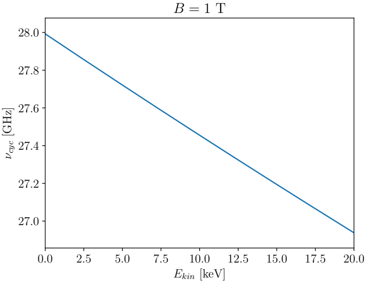

Entire decay spectrum covers a frequency range of about 1 GHz

Differentiate ν_{cyc} w.r.t. E_{tot} to give

$$\frac{d\nu_{\text{cyc}}}{dE_{\text{tot}}} = -\frac{eBc^2}{2\pi}\left(E_{\text{tot}}\right)^{-2}$$
$$= -3.2 \times 10^{23}\,\text{Hz}\,\text{J}^{-1} = -51\,\text{kHz}\,\text{eV}^{-1}\,.$$

Solution

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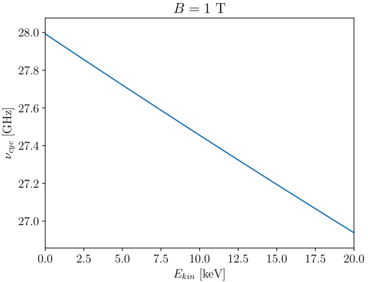

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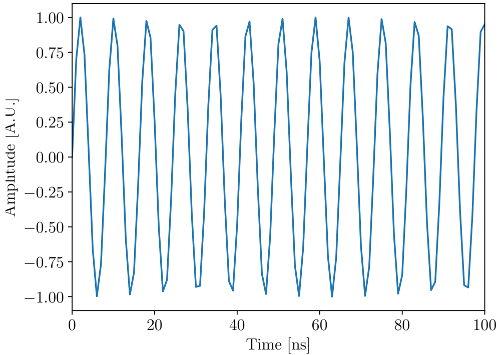
Therefore, at a minimum we require a bandwidth of 5 MHz to measure the last 100 eV of the decay spectrum.

CRES signals

To 1st order, our CRES signal is a monotonic sine wave.

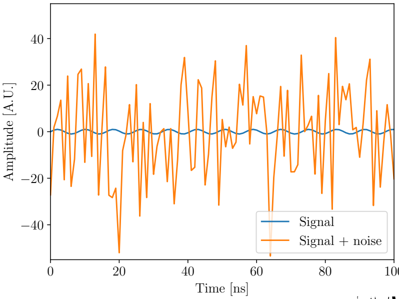


Adding noise...

We expect our primary noise contribution to be white (constant as a function of frequency).

If we add some white noise, our time series no longer looks like our signal.

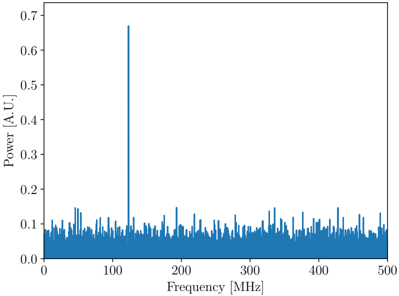
How can we use the features of our signal to discriminate from noise?



Solution

If we take the Fourier transform over a sufficient period of time, our signal is obvious in the frequency domain.

Signal is just a δ-function here!



How long do we need to observe our electrons for?

We want to measure our electron energies with a precision of $\Delta E = 1\,\text{eV}$ using a Fast Fourier Transform (FFT).

$$\frac{d\nu_{\text{cyc}}}{dE_{\text{tot}}} = -51\,\text{kHz}\,\text{eV}^{-1}$$

How long do we need to observe our electron for to achieve this?

Solution

To measure a width in energy $\Delta E = 1\,\text{eV}$, we need a width in frequency of

$$\Delta\nu = \left|\frac{d\nu_{\text{cyc}}}{dE_{\text{tot}}}\right| \times \Delta E$$
$$= 51\,\text{kHz}\,.$$

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For an FFT, the width in frequency is given by

$$\Delta\nu \sim \frac{1}{t_{\text{obs}}}\,.$$

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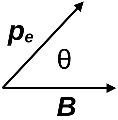
For our situation this gives us a required observation time of

$$t_{\text{obs}} \gtrsim 20\,\mu\text{s}\,.$$

What does this mean for our experiment design?

Solution

$$t_{\text{obs}} \gtrsim 20\,\mu\text{s}$$



An endpoint electron ($\beta = 0.263$) with a pitch angle of 89° will travel a distance parallel to the magnetic field of

$$vt\cos\theta = 0.263c \times \cos(89^\circ) \times 20\,\mu\text{s} = 275\,\text{m}\,.$$

Solution

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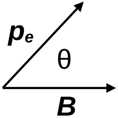

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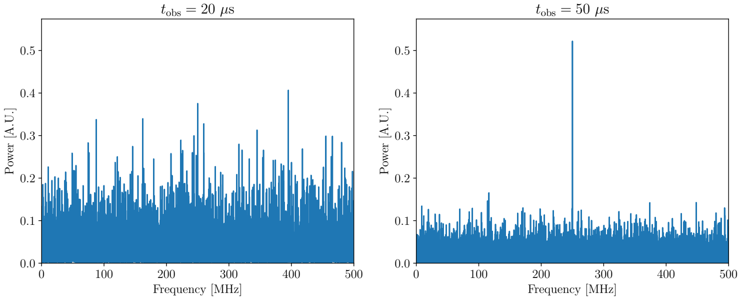
We need to confine our electrons so that they can be measured!

Optimising the frequency bin width

If using FFTs to measure our signal, increasing t_{obs} also has the potential to increase our signal-to-noise ratio.

$$\Delta\nu = \frac{1}{t_{\mathrm{obs}}}$$



Noise has constant power spectral density and signal can be approximated as δ-function

Optimising the frequency bin width

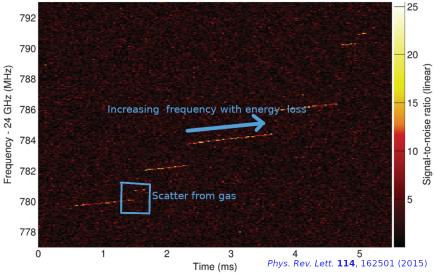
So... measuring electrons for longer with FFTs means better frequency precision and better SNR.

Are there any reasons we just can't measure for longer and longer?

Minimum frequency bin width

Are there any reasons we just can't measure for longer and longer?

1. Our electrons will eventually scatter off residual tritium gas and we will lose information about their initial energy
2. The electrons are radiating energy and their cyclotron frequency is changing



Phys. Rev. Lett. **114**, 162501 (2015)

Optimum frequency bin width

Putting aside concerns about scattering for the moment, is there a frequency bin width which maximises the signal-to-noise scenario?

$$\Delta\nu = \frac{1}{t_{\mathrm{obs}}}$$

$$\nu_{\mathrm{cyc}} = \frac{1}{2\pi}\frac{eBc^2}{E_{\mathrm{tot}}}$$

$$P_{\mathrm{Larmor}} = \frac{2\pi e^2 \nu_{\mathrm{cyc}}^2 \beta^2 \sin^2\theta}{3\epsilon_0 c}$$

Solution

The optimum frequency bin width occurs when the observation time required to produce a given frequency bin width is the same as the time it takes the electron's frequency to change by the same amount.

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Differentiate the cyclotron frequency equation w.r.t. time

$$\frac{d\nu_{\mathrm{cyc}}}{dt} = -\frac{eBc^2}{2\pi}\frac{1}{E_{\mathrm{tot}}}\frac{dE_{\mathrm{tot}}}{dt}$$

$$\frac{dE_{\mathrm{tot}}}{dt} = -P_{\mathrm{Larmor}}$$

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Expanding and simplifying gives:

$$\frac{d\nu_{\mathrm{cyc}}}{dt} = \frac{1}{E_{\mathrm{tot}}^2}\gamma^2\beta^2\sin^2\theta\left(\frac{e^5B^3c}{12\pi^2\epsilon_0 m_e^2}\right)$$

Solution

Frequency change from radiation over t_{obs} is $\frac{d\nu_{\mathrm{cyc}}}{dt} \times t_{\mathrm{obs}}$ and frequency bin width is $1/t_{\mathrm{obs}}$. Equate these two expressions for frequency and solve to find

$$t_{\mathrm{obs,opt}} = \left(\frac{d\nu_{\mathrm{cyc}}}{dt}\right)^{-\frac{1}{2}}$$

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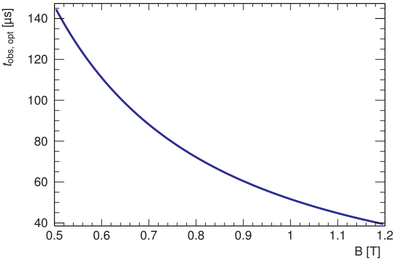
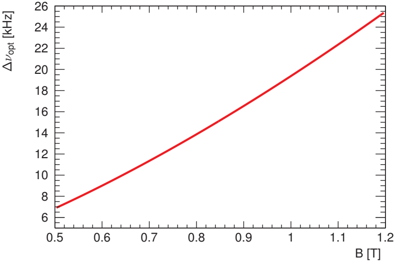
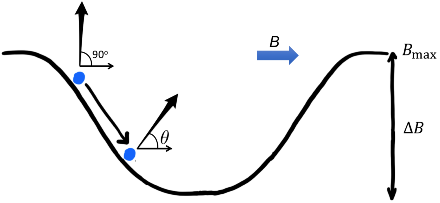




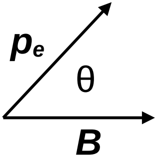
Trapping



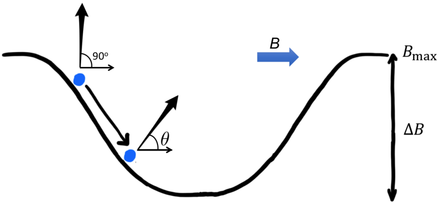
Phys. Rev. C **99**, 055501 (2019)

- Need to trap our electrons:
  - Achieve required energy and frequency resolution
  - Collect sufficient power
  - But don't change their energy!

- Trap electrons in 'no-work' magnetic trap where they can be observed, undergoing periodic motion
- Local minimum in magnitude of background *B*-field
- Only electrons with pitch angles above a certain value are trapped
- Trapped electrons climb up magnetic field potential until they eventually change direction



Which pitch angles do we trap?



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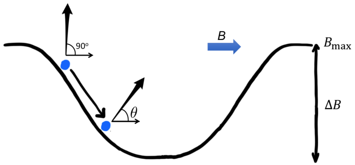
For a given magnetic field maximum, B_{max} and trap depth, ΔB, what is the minimum electron pitch angle we expect to trap?

Electron magnetic moment given by

$$\mu(t) = \frac{1}{2} \frac{p_0^2}{m_e} \frac{\sin^2 \theta(t)}{B(t)}$$

Adiabatic approximation – slowly changing B field means that μ is constant with time

Solution

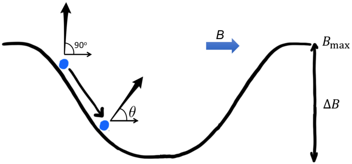


Phys. Rev. C **99**, 055501 (2019)

Divide electron KE into components $\parallel$ and $\perp$ to B-field

$$E_{\mathrm{kin}} = E_{\mathrm{kin}\parallel} + E_{\mathrm{kin}\perp}$$
$$= \frac{1}{2}\frac{p_0^2}{m_e}\cos^2\theta(t) + \mu(t)B(t)$$
$$\mu(t) = \frac{1}{2}\frac{p_0^2}{m_e}\frac{\sin^2\theta(t)}{B(t)}$$

Solution



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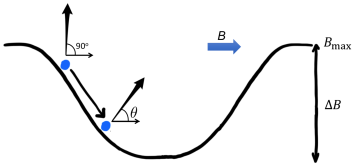
Recall that μ is constant with time (adiabatic)

At trap bottom: $\theta = \theta_{\rm bot}$, $B = B_{\rm max} - \Delta B$

Solution



Phys. Rev. C **99**, 055501 (2019)

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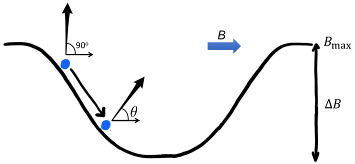
Recall that μ is constant with time (adiabatic)

At trap bottom: $\theta = \theta_{\mathrm{bot}}$, $B = B_{\max} - \Delta B$

For electrons that are *just* trapped: $\theta = \pi/2$, $B = B_{\max}$

Solution

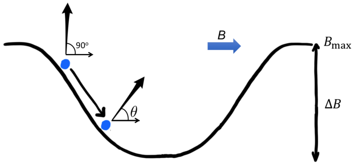


Phys. Rev. C **99**, 055501 (2019)

Equating expressions for μ at the top and bottom of the trap we find...

$$\frac{1}{2}\frac{p_0^2}{m_e}\frac{\sin^2\theta_{\mathrm{bot}}}{B_{\mathrm{max}}-\Delta B}=\frac{1}{2}\frac{p_0^2}{m_e}\frac{1}{B_{\mathrm{max}}}$$

Solution



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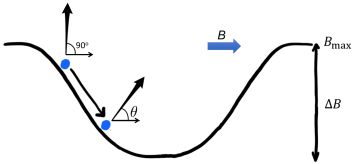
Rearranging, find that the pitch angle at the trap bottom for a just trapped electron is given by:

$$\theta_{\text{bot}}=\sin^{-1}\left(\sqrt{1-\frac{\Delta B}{B_{\text{max}}}}\right)$$

Solution



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Rearranging, find that the pitch angle at the trap bottom for a just trapped electron is given by:

$$\theta_{\mathrm{bot}}=\sin^{-1}\left(\sqrt{1-\frac{\Delta B}{B_{\max}}}\right)$$

Therefore trapping condition given by:

$$\theta_{\mathrm{bot}}\geq\sin^{-1}\left(\sqrt{1-\frac{\Delta B}{B_{\max}}}\right)$$

Parameter estimation

If our signal, embedded in white noise (variance σ^2) is sampled at N discrete times, each separated by at time T, with what precision can the frequency be measured?

Fisher information matrix for a deterministic signal (unknown parameters $\vec{\theta}$) in Gaussian white noise is given by:

$$\mathbf{I}(\vec{\theta}) = \frac{1}{\sigma^2}\mathrm{Re}\left[\frac{\partial \vec{s}(t|\vec{\theta})}{\partial \vec{\theta}} \cdot \frac{\partial \vec{s}^{\dagger}(t|\vec{\theta})}{\partial \vec{\theta}}\right]$$

Signal vector $\vec{s} = [s_0, s_1, ..., s_{N-1}]$ where we model our signal as a complex single tone wave:

$$s_n = Ae^{i(\omega t_n + \phi_0)}$$

and $t_n = t_0 + nT = (n_0 + n)T$.

Solution

We can calculate that:

$$s_n s_n^* = A^2$$
$$\vec{s} \cdot \vec{s}^\dagger = NA^2$$

Differentiating $\vec{s}$ w.r.t. unknown parameters gives

$$\frac{\partial \vec{s}}{\partial A} = e^{i(\omega \vec{t} + \phi_0)} = \frac{1}{A}\vec{s}$$
$$\frac{\partial \vec{s}}{\partial \omega} = iA\vec{t}e^{i(\omega \vec{t} + \phi_0)} = i\vec{t}\vec{s}$$
$$\frac{\partial \vec{s}}{\partial \phi_0} = iAe^{i(\omega \vec{t} + \phi_0)} = i\vec{s}$$

UCL

Solution

We can then calculate the elements of the Fisher information matrix:

$$I_{AA} = \frac{N}{\sigma^2} \qquad I_{\omega\omega} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} t_n^2 = \frac{A^2}{\sigma^2}\langle 2\rangle$$

$$I_{\phi_0\phi_0} = \frac{NA^2}{\sigma^2} \qquad I_{A\phi_0} = I_{\phi_0 A} = \frac{1}{A\sigma^2}\mathrm{Re}[i\vec{s}\cdot\vec{s}^\dagger] = 0$$

$$I_{A\omega} = I_{\omega A} = 0 \qquad I_{\omega\phi_0} = I_{\phi_0\omega} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} t_n = \frac{A^2}{\sigma^2}\langle 1\rangle$$

$$\mathbf{I}(\vec{\theta}) = \frac{A^2}{\sigma^2}\begin{bmatrix} \frac{N}{A^2} & 0 & 0 \\ 0 & \langle 2\rangle & \langle 1\rangle \\ 0 & \langle 1\rangle & N \end{bmatrix}$$

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Solution

$$\mathbf{I}^{-1} = \frac{\sigma^2}{A^2}\begin{bmatrix} \frac{A^2}{N} & 0 & 0 \\ 0 & -\frac{N}{\langle 1\rangle^2 - \langle 2\rangle N} & \frac{\langle 1\rangle}{\langle 1\rangle^2 - \langle 2\rangle N} \\ 0 & \frac{\langle 1\rangle}{\langle 1\rangle^2 - \langle 2\rangle N} & \frac{\langle 2\rangle}{\langle 1\rangle^2 - \langle 2\rangle N} \end{bmatrix}$$

where

$$\langle 1\rangle = \sum_{n=0}^{N-1} t_n = n_0 NT + \frac{N(N-1)T}{2}$$

$$\langle 2\rangle = \sum_{n=0}^{N-1} t_n^2 = \frac{NT^2}{6}\left[1 + 2N^2 + 6n_0(n_0-1) + N(6n_0-3)\right]$$

UCL

Solution

The Cramer-Rao lower bound for a parameter, θ_i, can be expressed as

$$\mathrm{var}\left(\hat{\theta}_i\right) \geq (\mathbf{I}^{-1})_{\theta_i\theta_i}\,.$$

For estimator of the angular frequency, we find that

$$\mathrm{var}\left(\hat{\omega}\right) \geq \frac{12\sigma^2}{A^2 N(N^2-1)T^2}$$

$$\mathrm{var}\left(\hat{\omega}\right) \gtrsim \frac{12\sigma^2}{A^2 N^3 T^2}\,.$$

Express this in detector units of sample rate, $\nu_\mathrm{s} = 1/T$, observation time, $t_\mathrm{obs} = NT$ and $\mathrm{SNR} = A^2/2\sigma^2$:

$$\mathrm{var}\left(\hat{\omega}\right) \gtrsim \frac{6}{\mathrm{SNR}\,t_\mathrm{obs}^3\nu_\mathrm{s}}\,.$$

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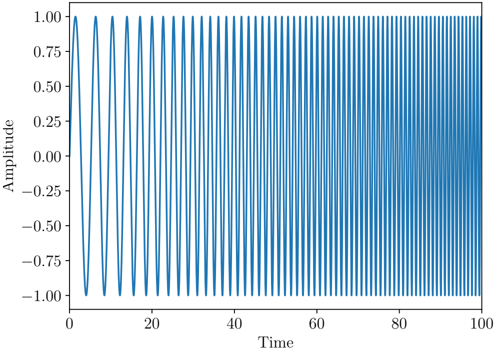
Chirping sine wave

Our signal frequency is constantly changing from the moment the electron is produced – signal better represented by:

$$s_n = A \exp\left[i\left(\omega_0 t_n + c t_n^2 + \phi_0\right)\right] \ .$$



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$$s_n = A \exp\left[i\left(\omega_0 t_n + c t_n^2 + \phi_0\right)\right] .$$

If we perform the same method as before, we now find

$$\mathrm{var}\left(\hat{\omega}_0\right) \gtrsim \frac{96}{\mathrm{SNR}\, t_{\mathrm{obs}}^3 \nu_{\mathrm{s}}}$$

which is $16\times$ larger than the previous result.

Frequency precision

$$\mathrm{var}\left(\hat{\omega}_0\right) \gtrsim \frac{96}{\mathrm{SNR}\, t_{\mathrm{obs}}^3\, \nu_s}$$

This equation tells us the key routes to improving our frequency (and energy) precision

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- Increase the rate (e.g. of our digitizer) at which we sample our data
- Improve our SNR – ↑ collected power, ↓ noise

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This equation tells us the key routes to improving our frequency (and energy) precision

- Increase the rate (e.g. of our digitizer) at which we sample our data
- Improve our SNR – $\uparrow$ collected power, $\downarrow$ noise
- Most fruitfully – observe our electrons for longer!

Required observation time

Previous FFT-based calculation found that we require $t_{\mathrm{obs}} \gtrsim 20\,\mu\mathrm{s}$ to achieve energy resolution of 1 eV.

Now use the Cramér-Rao lower bound to determine the required observation time to achieve $\sigma_E = 1\,\mathrm{eV}$.

$$\mathrm{var}\left(\hat{\omega}_0\right) \gtrsim \frac{96}{\mathrm{SNR}\, t_{\mathrm{obs}}^3\, \nu_s}$$

Assume:

- Sample rate, $\nu_s = 1\,\mathrm{GHz}$
- Noise temperature, $T_{\mathrm{noise}} = 5\,\mathrm{K}$
- We collect 10% of the total radiated power in a 1 T magnetic field

Solution

Linear start frequency resolution given by $\sigma_{\nu_0} = \sqrt{\mathrm{var}\left(\hat{\omega}_0\right)}/2\pi$ which can be rearranged to give

$$t_{\mathrm{obs}}^3 = \frac{24}{\pi^2 \mathrm{SNR}\, \nu_s \sigma_{\nu_0}^2}$$

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Therefore, $\mathrm{SNR} = \frac{10^{-16}\,\mathrm{W}}{1.38\times10^{-23}\,\mathrm{J\,K^{-1}}\cdot 5\,\mathrm{K}\cdot 500\times10^6\,\mathrm{s^{-1}}} = 0.0029$.

UCL

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$$t_{\mathrm{obs}} = \left(\frac{24}{\pi^2 \cdot 0.0029 \cdot 10^6\,\mathrm{s^{-1}} \cdot (50 \times 10^3\,\mathrm{s^{-1}})^2}\right)^{1/3} \approx 70\,\mu\mathrm{s}$$

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Scattering constraints

From the previous example, we have a required observation time, $t_{\mathrm{obs}} \approx 70\,\mu\mathrm{s}$.

What kind of constraints does this put on our tritium atom number density?

Tritium scattering cross-section, $\sigma_0 \approx 9 \times 10^{-19}\,\mathrm{cm}^2$

For an endpoint electron, $\beta \approx 0.26$.

Solution

Say we have a tritium atom number density, n. The mean free path of an electron is given by

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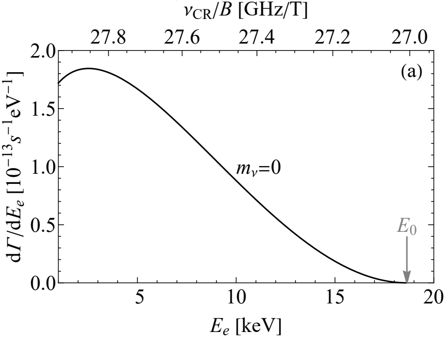
If we want our electrons' mean free time to be similar to our required observation time then we require

$$n = \frac{1}{\sigma_0 \beta c t_{\mathrm{obs}}} = \frac{1}{9 \times 10^{-23}\,\mathrm{m}^2 \cdot 0.26 \cdot 3 \times 10^{8}\,\mathrm{m\,s}^{-1} \cdot 70 \times 10^{-6}\,\mathrm{s}}$$
$$= 2 \times 10^{18}\,\mathrm{m}^{-3}$$

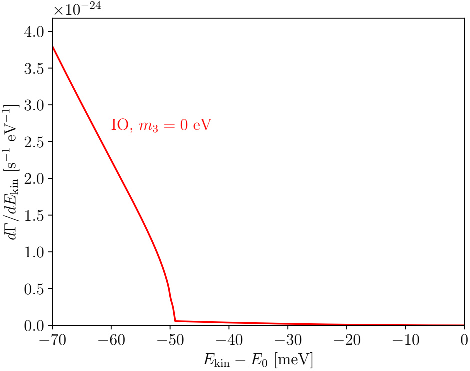
Reminder: Tritium β-decay spectrum



$$\frac{d\Gamma}{dE_{\mathrm{kin}}} \approx \frac{G_F \cos^2\theta_C}{2\pi^3}(g_V^2 + 3g_A^2)F(2, E_{\mathrm{kin}})|\mathbf{p}|(E_{\mathrm{kin}} + m_e)$$

$$\times \sum_{i=1}^{3} |U_{ei}|^2 (E_0 - E_{\mathrm{kin}})\sqrt{(E_0 - E_{\mathrm{kin}})^2 - m_i^2}\,\Theta(E_0 - E_{\mathrm{kin}} - m_i)$$

Behaviour close to spectrum endpoint



$$\frac{d\Gamma}{dE_{\mathrm{kin}}} \approx C \sum_{i=1}^{3} |U_{ei}|^2 (E_0 - E_{\mathrm{kin}}) \sqrt{(E_0 - E_{\mathrm{kin}})^2 - m_i^2}\, \Theta (E_0 - E_{\mathrm{kin}} - m_i)$$

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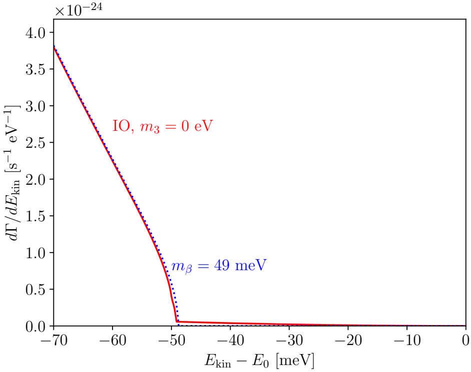

When we measure the decay spectrum we fit for m_β^2 where

$$m_\beta^2 = \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \, .$$

$$\frac{d\Gamma}{dE_{\mathrm{kin}}} \approx C\,(E_0 - E_{\mathrm{kin}})\sqrt{(E_0 - E_{\mathrm{kin}})^2 - m_\beta^2}\,\Theta\,(E_0 - E_{\mathrm{kin}} - m_\beta)$$

Basic sensitivity calculation: 1

For a background-free, systematics-free experiment, how many tritium atoms do we need to observe for 1 year to achieve a 90% CL limit on m_β of 0.1 eV?

$$\frac{dN}{dE} = 3rt(E_0 - E)\left[(E_0 - E)^2 - m_\beta^2\right]^{1/2}$$

- r: Rate in last eV of spectrum with $m_\beta = 0$
- t: Running time

Measure m_β using single measurement of N_{tot} events in energy interval, $\Delta E = E_0 - E_1$

Assume we can achieve $\Delta E = 1$ eV

Step 1: How many decays do we expect in our energy interval, ΔE?

Solution

$$
\begin{aligned}
N &= \int_{E_1}^{E_0} 3rt(E_0 - E)\left[(E_0 - E)^2 - m_\beta^2\right]^{1/2} dE \\
&= \left[-rt\left[(E_0 - E)^2 - m_\beta^2\right]^{3/2}\right]_{E_1}^{E_0} \\
&= rt\left[(E_0 - E_1)^2 - m_\beta^2\right]^{3/2} \\
&= rt\Delta E^3\left[1 - \frac{m_\beta^2}{\Delta E^2}\right]^{3/2}
\end{aligned}
$$

Solution

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&= rt\Delta E^3\left[1 - \frac{m_\beta^2}{\Delta E^2}\right]^{3/2}
\end{aligned}
$$

For the case where $\Delta E \gg m_\beta^2$:

$$
N \approx rt\Delta E^3\left(1 - \frac{3}{2}\frac{m_\beta^2}{\Delta E^2}\right) .
$$

Basic sensitivity calculation: 2

We now have an expression for the expected number of tritium decays in a window ΔE.

$$N \approx rt\Delta E^3 \left(1 - \frac{3}{2}\frac{m_\beta^2}{\Delta E^2}\right)$$

Step 2: Calculate the variance of m_β^2

Solution

$$N \approx rt\Delta E^3 \left(1 - \frac{3}{2}\frac{m_\beta^2}{\Delta E^2}\right)$$

The variance of N may be expressed as

$$(\sigma_N)^2 = \left(\frac{\partial N}{\partial m_\beta^2}\right)^2 \left(\sigma_{m_\beta^2}\right)^2$$

UCL

Solution

$$N \approx rt\Delta E^3 \left(1 - \frac{3}{2}\frac{m_\beta^2}{\Delta E^2}\right)$$

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$$(\sigma_N)^2 = \left(\frac{\partial N}{\partial m_\beta^2}\right)^2 \left(\sigma_{m_\beta^2}\right)^2 \qquad \text{where} \qquad \frac{\partial N}{\partial m_\beta^2} = -\frac{3rt\Delta E}{2}.$$

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Solution

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Therefore,

$$\sigma_{m_\beta^2} = \frac{2}{3rt\Delta E}\sqrt{N} = \frac{2}{3rt\Delta E}\sqrt{rt\Delta E^3 \left(1 - \frac{3}{2}\frac{m_\beta^2}{\Delta E^2}\right)}$$

UCL

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$$\approx \frac{2}{3}\sqrt{\frac{\Delta E}{rt}}$$

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Basic sensitivity calculation: 3

We have now calculated the variance of m_β^2:

$$\sigma_{m_\beta^2} \approx \frac{2}{3}\sqrt{\frac{\Delta E}{rt}}\,.$$

Step 3: Returning to our original question: with a 1 year running time, how many tritium atoms must we observe to obtain a limit of $m_\beta \lesssim 0.1$ eV (90% CL)? Assume a trapping and detection efficiency of 1.

Rate in the last eV, r, may be expressed as

$$r = \epsilon \frac{N_{\mathrm{atom}}}{\tau_m}\eta\,.$$

- ϵ: Trapping and detection efficiency
- $\eta = 2 \times 10^{-13}\,\mathrm{eV}^{-3}$: Fraction of events in last eV
- τ_m: Mean tritium lifetime
- $\tau_{1/2} = 12.32$ years

Solution

$$m_{\beta,\,90\mathrm{CL}} = \sqrt{1.28\sigma_{m_\beta^2}} = \frac{8}{5\sqrt{3}}\left(\frac{\Delta E}{rt}\right)^{1/4} .$$

Solution

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Given that $r = \epsilon \frac{N_{\mathrm{atom}}}{\tau_m}\eta$ and setting $\epsilon = 1$:

$$m_{\beta,\,90\mathrm{CL}} = \frac{8}{5\sqrt{3}}\left(\frac{\Delta E\,\tau_m}{t\,\eta\,N_{\mathrm{atom}}}\right)^{1/4}$$

Solution

$$m_{\beta,\,90\mathrm{CL}} = \sqrt{1.28\sigma_{m_\beta^2}} = \frac{8}{5\sqrt{3}}\left(\frac{\Delta E}{rt}\right)^{1/4} .$$

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$$m_{\beta,\,90\mathrm{CL}} = \frac{8}{5\sqrt{3}}\left(\frac{\Delta E\,\tau_m}{t\,\eta\,N_{\mathrm{atom}}}\right)^{1/4}$$

Rearranging we find

$$N_{\mathrm{atom}} = \left(\frac{8}{5\sqrt{3}}\right)^4 \frac{\Delta E \tau_m}{t\eta(m_{\beta,\,90\mathrm{CL}})^4}$$

Solution

$$m_{\beta,\,90\mathrm{CL}} = \sqrt{1.28\sigma_{m_\beta^2}} = \frac{8}{5\sqrt{3}}\left(\frac{\Delta E}{rt}\right)^{1/4}.$$

Given that $r = \epsilon \frac{N_{\mathrm{atom}}}{\tau_m}\eta$ and setting $\epsilon = 1$:

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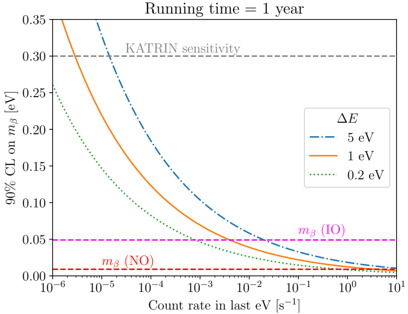
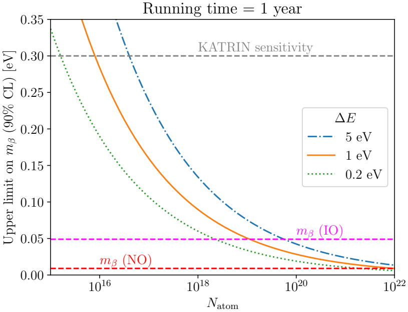
Rearranging we find

$$\begin{aligned} N_{\mathrm{atom}} &= \left(\frac{8}{5\sqrt{3}}\right)^4 \frac{\Delta E \tau_m}{t\eta (m_{\beta,\,90\mathrm{CL}})^4} \\ &= \left(\frac{8}{5\sqrt{3}}\right)^4 \frac{1\,\mathrm{eV}\cdot 5.6090\times 10^8\,\mathrm{s}}{3.1558\times 10^7\,\mathrm{s}\cdot 2\times 10^{-13}\,\mathrm{eV}^{-3}\cdot (0.1\,\mathrm{eV})^4} \\ &\approx 10^{18}\ \mathrm{atoms} \end{aligned}$$

Possible mass limits

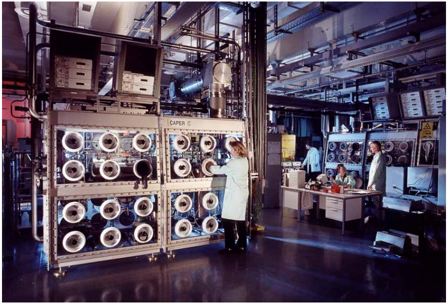




- Unfavourable sensitivity scaling with 'exposure' ($N_{\mathrm{atom}} \times \mathrm{time}$) – $(rt)^{-1/4}$
- 10^{18} atoms is a tritium mass of 5 µg – misleadingly small!
- Assumes all electrons are trapped and reco'd
- Need to maintain atomic tritium (rather than T_2) so require a larger inventory for regeneration

Tritium inventory

For comparison: TLK has an inventory of 25 g of T_2



Summary

- Hopefully this workshop has helped flesh out some of the details behind QTNM
- As you have seen, there are a number of competing factors which need to be addressed to make a CRES experiment work optimally
- Questions?